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SINGULARITIES OF THE BOUNDARIES OF STABILITY DOMAINS†

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The singularities of the general position which arise on the boundaries of stability domains of a linear, autonomous system of differential equations in a two- or three-dimensional parameter space are investigated. A constructive approach is proposed which enables one to determine the geometry of the singularities (the orientation in space, the magnitude of the angles, etc.) by constructing cones which are tangential to the stability domain using the first derivatives of the matrix operator of the system with respect to the parameters and its eigenvectors and associated vectors at the singular points of the boundary. Examples are presented. © 1999 Elsevier Science Ltd. All rights reserved.

A linear autonomous system of differential equations $\dot{y} = Ay$ where the matrix operator A depends smoothly on real parameters, is considered. The singularities of the general position, which arise on the boundaries of the stability domain in the two- or three-dimensional parameter space, are enumerated for these systems and they are described, apart from a smooth change in the parameters (diffeomorphism) [1].

A constructive approach is presented below which enables one to determine the geometry of the singularities by constructing cones that are tangential to the stability domain using the first derivatives of the matrix A with respect to the parameters and its eigenvectors and associated vectors at the singular points of the boundary. The method is based on the theory of perturbations of the eigenvalues of matrices which depend on parameters [2, 3] and the theory of the normal forms of families of matrices [1]. Problems on the stability of the equilibrium state in an electric arc circuit and the stability of the motion of a Ziegler double pendulum, loaded with a following force with two independent dissipation parameters are considered as examples. In the second problem, it is shown that the singularity which arises for the value of the critical force when no account is taken of dissipative forces, is "deadlock of an edge" according to the terminology of [1]. This singularity is reflected in the destabilization of the system by small dissipative forces and the absence of a critical load limit when the dissipation parameters tend to zero. The appearance of similar effects in the case of a singularity of a "break of an edge" type could also be expected.

1. DECOMPOSITION OF JORDAN BLOCKS

We consider the eigenvalue problem

$$Au = \lambda u \tag{1.1}$$

where A is an arbitrary real m-th order matrix, the elements of which $a_{ij}(p)$ $(i, j = 1, \dots, m)$ are smooth functions of the vector of the real parameters $p = (p_1, \dots, p_n)^T$, λ is an eigenvalue and u is an eigenvector of dimension m.

We shall consider the change in the eigenvalue as a function of the change in the parameters $p_1, \dots, p_n$. Suppose the number λ_0 is an eigenvalue of the matrix $A(p_0)$ when $p = p_0$. We now add the increment $p = p_0 + e\varepsilon + d(\varepsilon)\varepsilon^2$ to the vector of the parameters, where ε is a small positive number and e and $d(\varepsilon)$ are, respectively, an arbitrary, but fixed, direction vector and an arbitrary, but fixed, vector function which depends smoothly on ε. As a result, the matrix A is incremented, which is represented in the form of a series

$$A(p_0 + e\varepsilon + d(\varepsilon)\varepsilon^2) = A_0 + A_1\varepsilon + A_2\varepsilon^2 + \dots \tag{1.2}$$

$$\left(A_0 = A(p_0),\ A_1 = \sum_{s=1}^{n} \frac{\partial A}{\partial p_s} e_s \right)$$

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As a result of the perturbation of the parameter vector, the eigenvalue and the eigenvector are also incremented. According to the theory of perturbations of non-selfadjont operators [2], these increments have different representations depending on the Jordan structure of A_0.

1. In the case of the simple eigenvalue λ_0, the increments in the eigenvalue and the eigenvector are represented in the form of series in integral powers of ε

$$\lambda = \lambda_0 + \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \dots, \quad u = u_0 + w_1\varepsilon + w_2\varepsilon^2 + \dots \tag{1.3}$$

Together with the right eigenvector u_0, when $p = p_0$, we shall consider the left eigenvector v_0

$$v_0^T A_0 = \lambda_0 v_0^T \tag{1.4}$$

and use the normalization condition

$$v_0^T u_0 = 1 \tag{1.5}$$

We substitute the expansions (1.2) and (1.3) into (1.1) and use Eqs (1.4) and (1.5). As a result, we obtain expressions for the first correction [2, 3]

$$\lambda_1 = v_0^T A_1 u_0 = \sum_{s=1}^{n} \left(v_0^T \frac{\partial A}{\partial p_s} u_0 \right) e_s \tag{1.6}$$

On introducing the real, n-dimensional vectors r and k with components which are defined by the relations

$$r^s + ik^s = v_0^T \frac{\partial A}{\partial p_s} u_0, \quad s = 1, \dots, n \tag{1.7}$$

where i is the square root of -1, we write (1.6) in the form

$$\lambda_1 = (r, e) + i(k, e) \tag{1.8}$$

where the brackets denote a scalar product in $\mathbb{R}^n$. The vectors r and k are the gradients of the real and imaginary parts of the eigenvalue λ, calculated when $p = p_0$, respectively. It is then possible successively to determine the quantities w_1, λ_2, w_2, etc. from the equations of the perturbation method.

The complex conjugate quantities $\lambda_1 = (r, e) \pm i(k, e)$ correspond to the pair of complex conjugate eigenvalues $\lambda_0 = \alpha_0 \pm i\omega_0$, respectively. On taking account of (1.3) and (1.8) in the neighbourhood of the point p_0, the expression for the complex conjugate pair λ, $\bar{\lambda}$ takes the form

$$\lambda, \bar{\lambda} = \alpha_0 + (r, e)\varepsilon \pm i[\omega_0 + (k, e)\varepsilon] + o(\varepsilon) \tag{1.9}$$

If λ_0 is a real number, then the vector $k = 0$.

2. We will now consider the case of a repeated eigenvalue λ_0 with a second-order Jordan block. This means that, when $p = p_0$, the eigenvector u_0 and the associated vector u_1, which are determined from the equations

$$A_0 u_0 = \lambda_0 u_0, \quad A_0 u_1 = \lambda_0 u_1 + u_0 \tag{1.10}$$

correspond to the eigenvalue λ_0.

For the left eigenvector and associated vector v_0 and v_1, we have

$$v_0^T A_0 = \lambda_0 v_0^T, \quad v_1^T A_0 = \lambda_0 v_1^T + v_0^T \tag{1.11}$$

respectively.

It immediately follows from Eqs (1.10) and (1.11) that the vectors u_0, u_1, v_0, v_1 satisfy the conditions

$$v_0^T u_0 = 0, \quad v_0^T u_1 = v_1^T u_0 \tag{1.12}$$

As a result of the perturbation of the parameter vector $p = p_0 + e\varepsilon + d(\varepsilon)e^2$, the matrix A undergoes the incremental change (1.2). In the multiple case, the eigenvalue, generally speaking, decomposes into l simple eigenvalues which, together with the eigenvectors corresponding to them, are represented in the form of Newton–Puiseux series, containing terms with fractional powers $\varepsilon^{j/l}, j = 0, 1, 2, \ldots$, where l is the length of the Jordan chain [2]. In the case under consideration ($l = 2$), these expansions have the form

$$\lambda = \lambda_0 + \lambda_1\varepsilon^{1/2} + \lambda_2\varepsilon + \lambda_3\varepsilon^{3/2} + \ldots \tag{1.13}$$

$$u = u_0 + w_1\varepsilon^{1/2} + w_2\varepsilon + w_3\varepsilon^{3/2} + \ldots$$

Substituting expansion (1.13) and (1.2) into (1.1) and using relations (1.10)–(1.12), we obtain expressions for the first corrections λ_1 and λ_2 in the form

$$\lambda_1 = \pm\left(\frac{v_0^T A_1 u_0}{v_0^T u_1}\right)^{1/2} \tag{1.14}$$

$$\lambda_2 = \frac{v_0^T A_1 u_1 + v_1^T A_1 u_0 - \lambda_1^2 v_1^T u_1}{2v_0^T u_1} \tag{1.15}$$

Relations (1.13)–(1.15) hold if $v_0^T A_1 u_0 \neq 0$ (the "Γ" condition in [2]). Note that the eigenvectors u_0 and v_0 are defined apart from arbitrary factors while the associated vectors u_1, v_1 are defined apart from additive terms αu_0 and βv_0 respectively, where α and β are arbitrary constants. However, the quantities λ_1 and λ_2 in (1.14) and (1.15) are independent of this arbitrariness.

Assuming the vectors u_0 and u_1 to be fixed, we introduce the following normalization

$$v_0^T u_1 = 1, \quad v_1^T u_1 = 0 \tag{1.16}$$

which uniquely defines the vectors v_0 and v_1. Expressions (1.14) and (1.15) are simplified in this case. We introduce the vectors f_1, q_1, f_2, q_2 with components which are defined by the relations

$$f_1^s + iq_1^s = v_0^T \frac{\partial A}{\partial p_s} u_0, \quad s = 1, \ldots, n \tag{1.17}$$

$$f_2^s + iq_2^s = \frac{1}{2}\left(v_0^T \frac{\partial A}{\partial p_s} u_1 + v_1^T \frac{\partial A}{\partial p_s} u_0\right)$$

Using these vectors, the splitting of the double eigenvalue λ_0 is described by the expression

$$\lambda = \lambda_0 \pm \{[(f_1, e) + i(q_1, e)]\varepsilon\}^{1/2} + [(f_2, e) + i(q_2, e)]\varepsilon + o(\varepsilon) \tag{1.18}$$

If λ_0 is a real number, then the vectors $q_1 = q_2 = 0$.

We will now investigate the decomposition of the triple eigenvalue λ_0, corresponding to $p = p_0$, with a third-order Jordan block. The Jordan chain corresponding to this case has the form

$$A_0 u_0 = \lambda_0 u_0, \quad A_0 u_1 = \lambda_0 u_1 + u_0, \quad A_0 u_2 = \lambda_0 u_2 + u_1 \tag{1.19}$$

For the left eigenvector and the left associated vectors, we have

$$v_0^T A_0 = \lambda_0 v_0^T, \quad v_1^T A_0 = \lambda_0 v_1^T + v_0^T, \quad v_2^T A_0 = \lambda_0 v_2^T + v_1^T \tag{1.20}$$

respectively.

The vectors u_1, v_1 are related by the orthogonality conditions

$$v_0^T u_0 = v_0^T u_1 = v_1^T u_0 = 0 \tag{1.21}$$

and also by the equalities

$$v_1^T u_2 = v_2^T u_1, \quad v_0^T u_2 = v_1^T u_1 = v_2^T u_0 \tag{1.22}$$

These relations are easily proved by the direct use of the chains (1.19) and (1.20).

Note that the eigenvectors u_0 and v_0 are defined apart from arbitrary factors, associated vectors u_1, v_1 are defined apart from the additive terms $\alpha_1 u_0$ and $\beta_1 v_0$ and the vectors u_2, v_2 are defined apart from the additive terms $\alpha_1 u_1 + \alpha_2 u_0$ and $\beta_1 v_1 + \beta_2 v_0$ respectively, where α_i, β_i $(i = 1, 2)$ are arbitrary constants.

Assuming that the vectors u_0, u_1, u_2 are fixed, it is convenient to normalize the vectors v_0, v_1, v_2 as follows:

$$v_0^T u_2 = 1, \quad v_1^T u_2 = v_2^T u_2 = 0 \tag{1.23}$$

This normalization uniquely defines the sequence of vectors v_0, v_1, v_2.

We will now describe the decomposition of the eigenvalue λ_0 into three simple eigenvalues. In this case, the expansions of the eigenvalues and the eigenvectors corresponding to them have the form [2]

$$\lambda = \lambda_0 + \lambda_1\varepsilon^{1/3} + \lambda_2\varepsilon^{2/3} + \lambda_3\varepsilon + \ldots \tag{1.24}$$
$$u = u_0 + w_1\varepsilon^{1/3} + w_2\varepsilon^{2/3} + w_3\varepsilon + \ldots$$

Substituting expansions (1.24) and (1.2) into Eq. (1.1) and equating the coefficients of like powers of ε, we obtain the equations in the unknowns $\lambda_1, \lambda_2, \ldots$ and $w_1, w_2, \ldots$. Using (1.19)–(1.23), we find the first three coefficients λ_1 (i = 1, 2, 3) from these equations

$$\lambda_1 = \left(v_0^T A_1 u_0\right)^{1/3}, \quad \lambda_2 = \frac{v_0^T A_1 u_1 + v_1^T A_1 u_0}{3\lambda_1} \tag{1.25}$$
$$\lambda_3 = \frac{1}{3}\left(v_0^T A_1 u_2 + v_1^T A_1 u_1 + v_2^T A_1 u_0\right)$$

These expressions hold subject to the condition that $v_0^T A_1 u_0 \neq 0$ this case, the first relation of (1.25) defines three different complex values, after which λ_2 and λ_3 are uniquely defined.

We now introduce the vectors h_i and t_i (i = 1, 2, 3) with components defined by the relations

$$h_1^s + it_1^s = v_0^T \frac{\partial A}{\partial p_s} u_0$$
$$h_2^s + it_2^s = \left(v_0^T \frac{\partial A}{\partial p_s} u_1 + v_1^T \frac{\partial A}{\partial p_s} u_0\right), \quad s = 1, \ldots, n \tag{1.26}$$
$$h_3^s + it_3^s = \left(v_0^T \frac{\partial A}{\partial p_s} u_2 + v_1^T \frac{\partial A}{\partial p_s} u_1 + v_2^T \frac{\partial A}{\partial p_s} u_0\right)$$

Using these vectors, the decomposition of the third-order Jordan block in n-dimensional parameter space is described by the expression

$$\lambda = \lambda_0 + R\varepsilon^{1/3} + \frac{(h_2, e) + i(t_2, e)}{3R}\varepsilon^{2/3} + \frac{1}{3}[(h_3, e) + i(t_3, e)]\varepsilon + o(\varepsilon), \tag{1.27}$$
$$R = [(h_1, e) + i(t_1, e)]^{1/3}$$

where the cube root R has three different complex values. If λ_0 is a real number, the vectors $t_i = 0$ (i = 1, 2, 3).

2. ONE- AND TWO-PARAMETER FAMILIES OF MATRICES $A(p)$

Consider the linear autonomous system of differential equations

$$\dot{y} = Ay \tag{2.1}$$

with the matrix operator A. It is well known that the trivial solution $y \equiv 0$ of (2.1) is asymptotically stable if the real parts of all of the eigenvalues of matrix A are negative. If there is just a single eigenvalue λ for which $\mathrm{Re}\lambda > 0$ then the system is unstable. The case when $\mathrm{Re}\lambda = 0$ for certain eigenvalues and $\mathrm{Re}\lambda < 0$ for all the remaining eigenvalues corresponds to a boundary of the stability domain (BSD).

We will consider a single-parameter family of matrices $A(p)$, $p \in \mathbb{R}$. The boundary of the stability domain in the case of a general position is characterized by a simple eigenvalue $\lambda = 0$ and a pair of simple complex conjugate eigenvalues $\lambda = \pm i\omega$ [1]. In the technical literature, these cases are referred to as divergence and flutter respectively.

According to (1.9), in the case of simple eigenvalues λ in the neighbourhood of the BSD ($\mathrm{Re}\lambda = 0$ when $p = p_0$) we have

$$\mathrm{Re}\lambda = r(p - p_0) + o(|p - p_0|)$$

Stability and instability are therefore determined by the sign of the quantity $r = \mathrm{Re}(\upsilon_0^T \partial A/\partial p u_0)$. For example, if $r > 0$, then, when $p < p_0$ (at least for values of p which are close to p_0) $\mathrm{Re}\lambda < 0$, and the system is asymptotically stable while, when $p > p_0$, it is unstable. Note that, in the case of the general position, $r \neq 0$.

In the treatment of a general two-parameter family of real matrices $A(p)$, $p \in \mathbb{R}^2$, the BSD consists of smooth curves corresponding to a simple null eigenvalue or a pair of simple pure imaginary eigenvalues $\pm i\omega$, which transversally intersect one another at their end points. The vector of the normal r to these curves is defined by relation (1.7), where the vectors u_0, υ_0 correspond to the eigenvalues $\lambda_0 = 0$ or $\lambda_0 = \pm i\omega$. It follows directly from (1.9) that the vector of the normal r lies in the instability domain. At the points of nonsmoothness of the BSD, the family of matrices A is characterized by the following Jordan structures (strata): $F_1(0^2)$, $F_2(0, \pm i\omega)$, $F_3(\pm i\omega_1, \pm i\omega_2)$, which respectively imply the existence of a double-zero eigenvalue with a second-order Jordan block, the existence of a simple zero eigenvalue and a pair of simple pure imaginary complex conjugate eigenvalues and, finally, the existence of two different pairs of pure imaginary complex conjugate eigenvalues [1].

Using the expansions (1.18), for the second-order Jordan block with $\lambda_0 = 0$, we have

$$\lambda = \pm\sqrt{(f_1, e)\varepsilon} + (f_2, e)\varepsilon + o(\varepsilon)$$

where the vectors f_1 and f_2 correspond to $\lambda_0 = 0$ and are calculated using (1.17). In the case of the general position, f_1 and f_2 are linearly independent.

If $e(f_1, e) < 0$ and $(f_2, e) < 0$ for a fixed direction, then we obtain that $\mathrm{Re}\lambda < 0$ (stability) in the case of sufficiently small ε. If, however, one of these inequalities has the opposite sign, then we have $\mathrm{Re}\,\lambda > 0$ (instability) for one of the eigenvalues in the case of sufficiently small ε.

In the subsequent reasoning we will make use of the concept of a tangential cone (TC) [4]. A tangential cone to a stability domain at a point on its boundary is the set of directions of the vectors along which a curve, which lies in the stability domain with the exception of the initial point, can be released from the given point. The TC is therefore a first approximation to the stability domain in the neighbourhood of a singular point. It is said to be non-degenerate if it cuts out a set of non-zero measure on the sphere. Otherwise, the TC is said to be degenerate.

The TC at the point of a BSD which corresponds to the stratum $F_1(0^2)$ is written in the form

$$K_{F1} = \{e : (f_1, e) \leqslant 0,\ \ (f_2, e) \leqslant 0\} \tag{2.2}$$

Using the expansions for simple eigenvalues (1.9), we construct the TC's at the points of the BSD corresponding to the strata $F_2(0, \pm i\omega)$, $F_3(\pm i\omega_1, \pm i\omega_2)$ in an analogous manner

$$K_{F2} = \{e : (r_0, e) \leqslant 0,\ \ (r, e) \leqslant 0\} \tag{2.3}$$
$$K_{F3} = \{e : (r_1, e) \leqslant 0,\ \ (r_2, e) \leqslant 0\}$$

where the vectors r_0, r, r_1, r_2 correspond to the simple eigenvalues 0 $\pm i\omega$, $\pm i\omega_1$, $\pm i\omega_2$, respectively. In the case of a general position, r_0 and r as well as r_1 and r_2 are linearly independent.

According to (2.2) and (2.3), the stability domain at the singular points F_1, F_2, F_3 is wedged into the domain of instability and the angle of the wedge is less than π (Fig. 1, the stability domain is shown hatched). This fact reflects the well-known principle of the "fragility of the good" [1, 5] and the property of quasiconvexity of the stability domain [4].

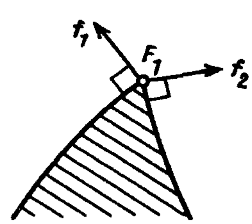

Fig. 1.

Example. We will now consider the problem of the stability of the equilibrium state in an electric arc circuit which is successively switched on with a resistance R, a self-inductance L and a shunted capacitance C [6]. The differential equations of the system, which have been linearized in the neighbourhood of the equilibrium state, have the form

$$\frac{d\xi}{dt}=-\frac{\rho\xi}{L}+\frac{\eta}{L},\quad \frac{d\eta}{dt}=-\frac{\xi}{C}-\frac{\eta}{RC} \tag{2.4}$$

where $\xi(t)$, $\eta(t)$ are the arc current and voltage and ρ is the resistance of the arc.

System (2.4) depends on four parameters: the three positive quantities L, C and R and the parameter ρ, which can take both positive and negative values. Assuming the parameters L and C to be fixed, we will investigate the stability of system (2.4) in the plane of the parameters R and ρ.

The characteristic equation of system (2.4) can be written in the form [6]

$$\lambda^2+\left(\frac{1}{RC}+\frac{\rho}{L}\right)\lambda+\frac{1}{LC}\left(\frac{\rho}{R}+1\right)=0 \tag{2.5}$$

At the point $R = R_*$, $\rho = -R_*$ ($R_* = \sqrt{(L/C)}$) the characteristic equation (2.5) has a double zero $\lambda_0 = 0$ which corresponds to a second-order Jordan block. Actually, using the notation $p_1 = R, p_2 = \rho$ and

$$A=\begin{Vmatrix}-\rho/L & 1/L\\ -1/C & -1/(RC)\end{Vmatrix} \tag{2.6}$$

we find, by formulae (1.10), (1.11) and (1.16), that

$$u_0=\begin{Vmatrix}1\\ -\sqrt{L/C}\end{Vmatrix},\quad u_1=\begin{Vmatrix}0\\ L\end{Vmatrix},\quad v_0=\begin{Vmatrix}1/\sqrt{LC}\\ 1/L\end{Vmatrix},\quad v_1=\begin{Vmatrix}1\\ 0\end{Vmatrix}$$

According to (2.7), using these vectors and the matrix A from (2.6) we obtain

$$f_1=-\frac{1}{L/\sqrt{LC}}\begin{Vmatrix}1\\ 1\end{Vmatrix},\quad f_2=\frac{1}{2L}\begin{Vmatrix}1\\ -1\end{Vmatrix}$$

and, hence, we find the TC (2.2) to the stability domain at the point $R = R_*$, $\rho = -R_*$. Its angle is equal to $\pi/2$ since the vectors f_1 and f_2 are orthogonal. This result agrees with that obtained earlier in [6]: the BSD is described by the straight line $\rho = -R$, $0 \leqslant R \leqslant R_*$ and the hyperbola $\rho = -L/(RC)$, $R_* \leqslant R$.

3. THREE-PARAMETER FAMILIES

Consider a smooth three-parameter family of real matrices $A(p), p \in \mathbb{R}^3$ in the general position. The BSD in the case of Eq. (2.1) in three-dimensional space is a smooth surface characterized by a single simple eigenvalue $\lambda = 0$ or a pair of simple pure imaginary eigenvalues $\lambda = \pm i\omega$ [1]. The vector of the normal to this surface r is defined, as in the two-dimensional case, by (1.7) and lies in the instability domain. The singularities of the BSD are comprehensively covered by the following list [1]: "dihedral angle", "trihedral angle", "deadlock of an edge" and "break of an edge".

A singularity of the "dihedral angle" type is associated with the strata $F_1(0^2)$, $F_2(0, \pm i\omega)$ and $F_3(\pm i\omega_1, \pm i\omega_2)$, which were considered in Section 2. The TC's in the stability domain for these singularities are determined by relations (2.2) and (2.3).

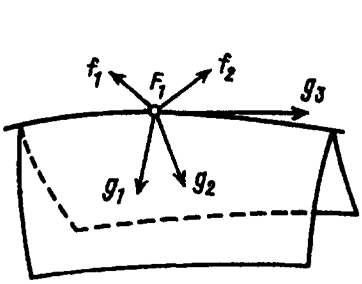

Fig. 2.

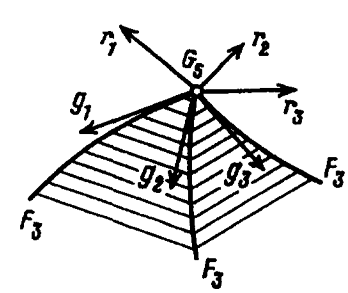

Fig. 3.

It is convenient to describe the TC K_{F1} by determining the vectors g_1, g_2, g_3 which satisfy the conditions

$$(g_i, f_j) = -\delta_{ij},\ i = 1, 2, 3,\ j = 1,2 \tag{3.1}$$

where δ_{ij} is the Kronecker delta and f_1, f_2 are the vectors from (2.2). Equations (3.1) are solvable for the vectors g_i since the vectors f_1, f_2 are linearly independent in the case of the general position.

The vector g_3 is directed along the edge of a "dihedral angle" and the vectors g_1 and g_2 are tangential to the sides of this angle (Fig. 2). Using the vectors g_1, g_2, g_3, the set (2.2) is described in the following manner in the three-dimensional case

$$K_{F1} = \{e : e = \alpha g_1 + \beta g_2 + \gamma g_3;\ \alpha, \beta, \gamma \in \mathbb{R},\ \alpha \geqslant 0,\ \beta \geqslant 0\} \tag{3.2}$$

Substituting the expression $e = \alpha g_1 + \beta g_2 + \gamma g_3$ into (2.2) and using (3.1), we find that $(f_1, e) = -\alpha \leqslant 0$, $(f_2, e) = -\beta \leqslant 0$, which proves representation (3.2). Similar representations for the TC's K_{F2} and K_{F3} can be obtained by using the vectors r_0, r and r_1, r_2 respectively in (3.1) instead of f_1, f_2.

A singularity of the "trihedral angle" type is characterized by the following strata: $G_3(0^2, \pm i\omega)$ is a double zero with a second-order Jordan block and a pair of simple pure imaginary complex conjugate eigenvalues, $G_4(0, \pm i\omega_1, \pm i\omega_2)$ is a simple zero and two pairs of different simple pure imaginary eigenvalues and $G_5(\pm i\omega_1, \pm i\omega_2, \pm i\omega_3)$ is three pairs of different pure imaginary eigenvalues [1]. Note that these structures differ from the strata F_1, F_2, F_3 considered above in the presence of an additional pair of the $\pm i\omega$ type. By analogy with (2.2) and (2.3), the TC's to the stability domain for these cases are therefore written in the form

$$\begin{aligned} K_{G3} &= \{e : (f_1, e) \leqslant 0,\ (f_2, e) \leqslant 0,\ (r, e) \leqslant 0\} \\ K_{G4} &= \{e : (r_0, e) \leqslant 0,\ (r_1, e) \leqslant 0,\ (r_2, e) \leqslant 0\} \\ K_{G5} &= \{e : (r_1, e) \leqslant 0,\ (r_2, e) \leqslant 0,\ (r_3, e) \leqslant 0\} \end{aligned} \tag{3.3}$$

where the vectors r and r_i correspond to the simple pairs $\pm i\omega$ and $\pm i\omega_1$ ($i = 1, 2, 3$) and are determined from relations (1.7), the vector r_0 corresponds to a simple zero and the vectors f_1 and f_2 correspond to a double zero with a second-order Jordan block and are found from (1.17).

The sets (3.3) are determined by three vectors and describe a trihedral angle (Fig. 3), which lies in a closed half-space. As in (3.1), it is possible to set up the vectors g_1, g_2 and g_3 in accordance with the three vectors defining the trihedral angle, r_1, r_2, r_3 in K_{G5}, for example, using the formulae

$$(g_i, r_j) = -\delta_{ij},\ i, j = 1, 2, 3$$

The vectors g_i are tangential to the edges of the trihedral angle. Using these vectors, the set K_{G5} is described as follows:

$$K_{G5} = \{e : e = \alpha g_1 + \beta g_2 + \gamma g_3;\ \alpha, \beta, \gamma \geqslant 0\} \tag{3.4}$$

Similar representations can also be obtained for K_{G3}, K_{G4}.

Note that the vectors defining the dihedral and trihedral angles are linearly independent in the case of the general position.

A "deadlock of an edge" singularity is characterized by a stratum $G_2((\pm i\omega)^2)$, that is, by the presence of a pair of double pure imaginary complex conjugate eigenvalues with second-order Jordan blocks. It is well known that the stability domain in the neighbourhood of this singularity, apart from a smooth change of coordinates (diffeomorphism), has the form [1]

$$z + |\operatorname{Re}\sqrt{x + iy}| < 0 \tag{3.5}$$

The TC to the stability domain (3.5) at the singular point of the boundary $G_2(x = y = z = 0)$ is degenerate and is the plane angle

$$K_{G2}^0 = \{e = (e_1, e_2, e_3) : e_1 \leqslant 0,\ e_2 = 0,\ e_3 \leqslant 0\}$$

Note that the singularity G_2, which is a "deadlock" of the edge F_3, is formed when two simple eigenvalues $i\omega_1$ and $i\omega_2$, corresponding to F_3, collide.

We will now calculate the TC for this singularity in the general case. Using the expansions (1.18) for the second-order Jordan block with $\lambda_0 = i\omega$, we have

$$\lambda = i\omega \pm [(f_1,e)+i(q_1,e)]^{1/2}\varepsilon^{1/2} + [(f_2,e)+i(q_2,e)]\varepsilon + o(\varepsilon) \tag{3.6}$$

where the vectors f_1, q_1, f_2, q_2 correspond to $\lambda_0 = i\omega$ and are calculated using (1.17). An expansion for the complex conjugate quantity with $\lambda_0 = -i\omega$ can be obtained by taking the complex conjugate in (3.6). If $(q_1, e) \neq 0$ or $(f_1, e) > 0$ in the expression under the square root sign in (3.6), then one of the eigenvalues (3.6) has a positive real part (instability) for sufficiently small ε. In the case when $(f_1, e) \leqslant 0$, $(q_1, e) = 0$, the second term of the expansion is a pure imaginary number and, consequently, when $(f_2, e) < 0$ and for sufficiently small ε, we have $\operatorname{Re}\lambda < 0$ (stability) and, when $(f_2, e) > 0$, we obtain $\operatorname{Re}\lambda > 0$ (instability). Hence, the TC to the stability domain at the singular point of the boundary G_2 is a plane angle and has the form

$$K_{G2} = \{e : (f_1,e) \leqslant 0,\ (f_2,e) \leqslant 0,\ (q_1,e)=0\} \tag{3.7}$$

By determining the vectors g_1, g_2 which satisfy the conditions

$$(g_i, f_j) = -\delta_{ij},\ \ (g_i, q_1) = 0,\ \ i,j = 1,2$$

we write the set (3.7) in the following manner

$$K_{G2} = \{e : e = \alpha g_1 + \beta g_2;\ \ \alpha,\beta \geqslant 0\} \tag{3.8}$$

where g_1 and g_2 are directed along the sides of the plane angle K_{G2} and g_1 is a vector which is tangential to the edge F_3 of the stability domain (Fig. 4).

Note that the vectors f_1, f_2, g_1 are linearly independent in the case of the general position.

4. BREAK OF AN EDGE

A singularity of the "break of an edge" type is characterized by a stratrum $G_1(0^3)$ with a triple zero eigenvalues of the operator $A(p_0)$ with Jordan chain (1.19). We now construct a versal deformation of the matrix $A_0 = A(p_0)$. It is determined by the family of matrices $A'(p')$, which depend smoothly on the vector of the parameters such that any smooth family $A(p)$, $p \in \mathbb{R}^3$ $(A(p_0) = A_0)$ in the neighbourhood of $p = p_0$ can be represented in the form [1]

$$A(p) = C(p)A'(\varphi(p))C^{-1}(p) \tag{4.1}$$

where $C(p)$ is a family of non-degenerate matrices which depend smoothly on p and $p' = \varphi(p)$ is a smooth mapping from the neighbourhood of the point p_0 in the space $\mathbb{R}^3$ into the neighbourhood of the origin of coordinates of the space $p' \in \mathbb{R}^d$, $\varphi_1(p_0) = \ldots = \varphi_d(p_0) = 0$. A versal deformation with the minimum possible number of parameters $p'_1, \ldots, p'_d$ is called a miniversal deformation. A miniversal deformation of the matrix A_0 can be chosen to be equal to a block-diagonal family of the form [1]

$$A'(p') = A'(0) + B(p') \tag{4.2}$$

In this expression, A'_0 is the Jordan upper triangular matrix of the operator A_0 and $B(p')$ is a block-diagonal matrix, the blocks of which correspond to the eigenvalues of the matrix A_0. The first block of the matrix $A'(p')$, which corresponds to the triple zero (0^3), can be chosen in the form

$$D_1(p') = \begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix} + \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p'_1 & p'_2 & p'_3 \end{Vmatrix} \tag{4.3}$$

and the other blocks correspond to the eigenvalues of the matrix A_0 with negative real part.

By virtue of relation (4.1), the characteristic equations for the matrices $A(p)$ and $A'(p')$, $p' = \varphi(p)$ are identical. By virtue of its block-diagonal structure, the stability of the matrix $A'(p')$ in the neighbourhood of the point $p' = 0$ is determined by the first block (4.3). The characteristic equation for it has the form $\lambda^3 - p'_3\lambda^2 - p'_2\lambda - p'_1 = 0$. The stability domain in the space of the parameters p'_1, p'_2, p'_3 is determined using the Routh–Hurwitz conditions

$$p'_1 + p'_2 p'_3 > 0,\ \ p'_1 < 0,\ \ p'_2 < 0,\ \ p'_3 < 0 \tag{4.4}$$

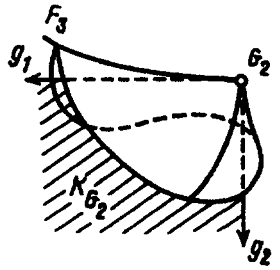

Fig. 4.

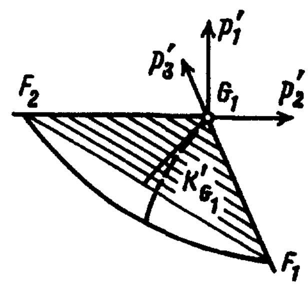

Fig. 5.

This domain is represented in Fig. 5. We immediately find from (4.4) that the cone which is tangential to the stability domain of the family $A'(p')$ at the point $p' = 0$ is degenerate and is defined by the relations

$$e_1' = 0, \quad e_2' \leqslant 0, \quad e_3' \leqslant 0 \tag{4.5}$$

We will calculate the vectors h_i' ($i = 1, 2, 3$), which describe the decomposition of the third-order Jordan block $D_1(0)$, by finding the eigenvectors and associated vectors u_i' and v_i' ($i = 0, 1, 2$) when $p' = 0$ and using formula (1.26). As a result, we obtain

$$h_1' = (1,0,0,\ldots 0)^T, \quad h_2' = (0,1,0,\ldots,0)^T, \quad h_3' = (0,0,1,0,\ldots,0)^T$$

Using these vectors, the TC (4.5) can be written in the form

$$K_{G1}' = \{e' = (e_1',\ldots,e_d'): \ (h_1',e') = 0, \ (h_2',e') \leqslant 0, \ (h_3',e') \leqslant 0\} \tag{4.6}$$

We will now determine the TC for the family $A(p)$. For this purpose, we find the relation between the vectors h_j' and h_j ($j = 1, 2, 3$). Suppose u_i', v_i' ($i = 0, 1, 2$) are the right and left eigenvectors and associated vectors of the matrix $A'(0)$, corresponding to the triple eigenvalue $\lambda_0 = 0$ and which satisfy the normalization conditions (1.23). Then, using (4.1), we find that the eigenvectors and associated vectors u_i, v_i ($i = 0, 1, 2$) of the matrix A_0 are related with u_i', v_i' as follows:

$$u_i = C(p_0)u_i', \quad v_i^T = v_i'^T C^{-1}(p_0), \quad i = 0,1,2 \tag{4.7}$$

We differentiate expression (4.1) with respect to p_i and find the value of the derivative when $p = p_0$, $p' = \varphi(p_0) = 0$

$$\frac{\partial A}{\partial p_i} = \frac{\partial C}{\partial p_i}A'C^{-1} + CA'\frac{\partial C^{-1}}{\partial p_i} + \sum_{j=1}^{d}\left(C\frac{\partial A'}{\partial p_j'}C^{-1}\right)\frac{\partial \varphi_j}{\partial p_i}, \quad i = 1,2,3 \tag{4.8}$$

We multiply both sides of equality (4.8) on the left by v_0^T and on the right by u_0. As a result, we have

$$h_1^i = v_0^T\frac{\partial A}{\partial p_i}u_0 = \sum_{j=1}^{d}\frac{\partial \varphi_j}{\partial p_i}\left(v_0'^T\frac{\partial A'}{\partial p_j'}u_0'\right), \quad i = 1,2,3 \tag{4.9}$$

Here, relations (4.7) and $A'(0)u'_0 = 0$, $v_0'^T A'(0) = 0$ have been used. From (4.9), we therefore obtain the relation between the vectors h_1 and h_1'

$$h_1^T = h_1'^T[\partial\varphi/\partial p]; \quad [\partial\varphi/\partial p] = [\partial\varphi_i/\partial p_j], \quad i = 1,\ldots,d, \quad j = 1,2,3$$

The fact that this relation also holds in the case of the vectors h_2 and h_3 can be proved in a similar way. Expressions (4.7) and (4.8) and the identities

$$v_s^T\frac{\partial C}{\partial p_i}C^{-1}u_j + v_s^T C\frac{\partial C^{-1}}{\partial p_i}u_j = v_s^T\frac{\partial}{\partial p_i}(CC^{-1})u_j = 0, \quad s,j = 0,1$$

are used in the proof.

Thus, the vectors h_s and h'_s are related by the equation

$$h_s^T = h_s'^T[\partial\varphi/\partial p], \quad s=1,2,3 \tag{4.10}$$

We now find the relation between the vectors of the directions e and e' from the TC in the spaces $\mathbb{R}^3$ and $\mathbb{R}^d$, respectively

$$e_i' = \frac{dp_i'}{d\varepsilon} = \sum_{j=1}^{d}\frac{\partial\varphi_i}{\partial p_j}\frac{dp_j}{d\varepsilon} = \sum_{j=1}^{d}\frac{\partial\varphi_i}{\partial p_j}e_j, \quad i=1,\dots,d, \quad j=1,2,3$$

Consequently

$$e' = [\partial\varphi/\partial p]e \tag{4.11}$$

Multiplying both sides of equality (4.1) by e and using (4.11), we obtain

$$h_s^T e = h_s'^T e', \quad s=1,2,3 \tag{4.12}$$

Using (4.6) and (4.12), we find the TC to the stability domain for the singularity $G_1(0^3)$ in the form

$$K_{G1} = \{e:(h_1,e)=0, \ (h_2,e)\leqslant 0, \ (h_3,e)\leqslant 0\} \tag{4.13}$$

The TC K_{G1} is degenerate and is a plane angle. We recall that the vectors h_1, h_2, h_3, which define it, are calculated from formula (1.26) using the eigenvectors and associated vectors corresponding to the triple zero. The vectors h_1, h_2, h_3 are linearly independent in the case of the general position.

Note that the RC lies in a plane in which the expansion (1.27) of the triple zero in powers of $\varepsilon^{1/3}$ does not hold, since $v_0^T A_1 u_0 = (h_1, e) = 0$ in this plane. Hence, in this case, the TC cannot be found by the method which was used to study other singularities. For this reason, an approach connected with the method of normal forms has been used here.

5. EXAMPLE

Consider a Ziegler double pendulum with two independent dissipation parameters [7]. It is a system with two degrees of freedom φ_1 and φ_2, consisting of two weightless sections of equal length l carrying point masses $m_1 = 2m$ and $m_2 = m$ and loaded at the free end with a following force P. It is assumed that the hinges of the system are viscoelastic and that moments

$$M_1 = k\varphi_1 + c_1\dot\varphi_1, \quad M_2 = k(\varphi_2-\varphi_1)+c_2(\dot\varphi_2-\dot\varphi_1)$$

arise in them.

The constants k, c_1, c_2 characterize the elastic and dissipative properties of the hinges. The linearized equations of the oscillations of the pendulum about the equilibrium position $\varphi_1 = \varphi_2 = 0$ have the form [7]

$$\begin{aligned}&3\ddot\varphi_1+\ddot\varphi_2+(\gamma_1+\gamma_2)\dot\varphi_1-\gamma_2\dot\varphi_2+(2-p)\varphi_1+(p-1)\varphi_2=0\\&\ddot\varphi_1+\ddot\varphi_2-\gamma_2\dot\varphi_1+\gamma_2\dot\varphi_2-\varphi_1+\varphi_2=0\end{aligned} \tag{5.1}$$

where the following dimensionless quantities have been used: the dissipation parameters $\gamma_1 = c_1/\sqrt{(kml^2)}$, $\gamma_2 = c_2/\sqrt{(kml^2)}$, the force $p = Pl/k$ and the time $\tau = t\sqrt{(k/ml^2)}$. By introducing the variables $\varphi_3 = \dot\varphi_1$ and $\varphi_4 = \dot\varphi_2$, we can write Eqs (5.1) in the form

$$\dot\varphi = A\varphi, \quad \varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T \tag{5.2}$$

$$A = \left\|\begin{matrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ p/2-3/2 & 1-p/2 & -\gamma_1/2-\gamma_2 & \gamma_2\\ 5/2-p/2 & p/2-2 & \gamma_1/2+2\gamma_2 & -2\gamma_2\end{matrix}\right\| \tag{5.3}$$

We shall investigate the singularities of the BSD of system (5.2), (5.3) in the space of the three parameters γ_1, γ_2, p. The roots of the characteristic equation of system (5.2), (5.3) when $\gamma_1 = \gamma_2 = 0$ are determined by the expression [7, 8]

$$\lambda^2 = (p-7/2\pm\Delta)/2, \quad \Delta = \sqrt{(p-7/2)^2-2} \tag{5.4}$$

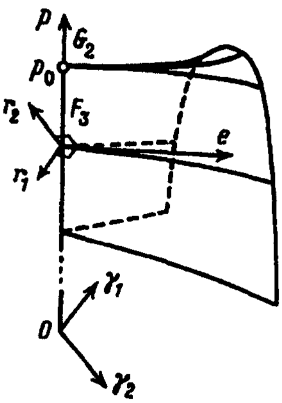

Fig. 6.

Consequently, when $p \in [0, p_0)$ ($p_0 = 7/2 - \sqrt{2}$), there are two different pairs of complex conjugate imaginary eigenvalues which correspond to a "dihedral angle" (F_3) singularity. A pair of complex conjugate imaginary eigenvalues with a Jordan chain (1.10) corresponds to the value p_0 which implies a "deadlock of an edge" (G_2) singularity. Hence, the segment

$$\gamma_1 = \gamma_2 = 0, \quad p \in [0, p_0] \tag{5.5}$$

is an edge of the BSD with a deadlock at the point $p = p_0$ (Fig. 6).

At an internal point of the segment (5.5), the TC $K_{F3}(p)$ is determined from (2.3). The vectors r_1 and r_2, calculated using formulae (1.7) for the matrix A from (5.3) are equal to

$$r_{1,2}(p) = (\pm(3-2p)/(16\Delta) - 1/8, \ \pm(19-6p)/(8\Delta) - 3/4, 0)^T \tag{5.6}$$

where the plus sign refers to r_1 and the minus sign refers to r_2. As p increases from zero, the angle between the vectors r_1 and r_2 (equal to the difference between π and the magnitude of the "dihedral angle") increases and, when $p \to p_0$, it attains a value of π and the moduli of the vectors r_1 and r_2 tend to infinity, since $\Delta = 0$ when $p = p_0$. Hence, the TC is degenerate, changing into the cone K_{G2} of a "deadlock of an edge" at the point $p = p_0$ (Fig. 6). The TC K_{G2} is defined in (3.7), where the vectors f_1, q_1, f_2 are calculated using formulae (1.17) and, in the case of the matrix (5.3), take the form

$$f_1 = (0, 0, 1)^T, \quad q_1 = (1, -4, -5\sqrt{2}, 0)^T, \quad f_2 = (-1, -6, 0)^T$$

apart from a positive factor.

This TC can be written in the form

$$K_{G2} = \{(e_1, e_2, e_3) : e_1 = (4 + 5\sqrt{2})e_2, \ e_2 \geqslant 0, \ e_3 \leqslant 0\} \tag{5.7}$$

In the space of the parameters (γ_1, γ_2, p), it is a plane angle.

For each fixed value of the parameters (γ_1, γ_2), the critical value of the load p_{cr} is determined as the smallest value of p for which the system becomes unstable. We consider the dissipation parameters in the form $\gamma_1 = e_1\varepsilon$, $\gamma_2 = e_2\varepsilon$, where ε is a small positive number. Since the segment (5.5) is an edge of the BSD, the critical loading limit $p_0^e = \lim p_{cr}(\gamma_1, \gamma_2)$ when $\varepsilon \to 0$ for a fixed direction (e_1, e_2) is equal to the value of p at which the vector $e = (e_1, e_2, 0)$ emerges from the TC $K_{F3}(p)$ (as p increases from zero). In this case, the condition $(r_1(p_0^e), e) = 0$ or $(r_2(p_0^e), e) = 0$ is satisfied. For example, when $\gamma_1 = \varepsilon$, $\gamma_2 = 0$, we have $e = (1, 0, 0)$, $p_0^e = 2$, $r_2(2) = (0, -120, 0)$, $(r_2(2), e) = 0$. It is clear from this that the value of the critical loading limit p_0^e is different for different directions (e_1, e_2). This limit is less than p_0 for all $(e_1, e_2) \neq c(4 + 5\sqrt{(2)}, 1)$ $(c > 0)$. When $(e_1, e_2) = c(4 + 5\sqrt{(2)}, 1)$, we have $p_0^e = p_0$. This is related to the fact that the directions $(4 + 5\sqrt{(2)}, 1, \alpha)$ when $\alpha \leqslant 0$ belong to the TC K_{G2} from (5.8).

The closing of the "dihedral angle" at the point of a "deadlock of an edge" singularity geometrically interprets the phenomenon of the destabilization of a non-conservative system by small dissipative forces and the indeterminacy of the critical load [7, 8]. We could expect similar effects to occur in the case of other systems with singularities in the BSD of a "deadlock of an edge" type and a "break of an edge" type.

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